

Analysis of inter-quark interactions in classical chromodynamics

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Abstract

The QCD gluon equation of motion is solved approximately by means of the Green function. This solution is used to reformulate the Lagrangian of QCD such that the gluon propagator appears directly in the interaction terms of the Lagrangian. The nature of the interactions is discussed. Their coordinate-space form is presented and analyzed in the static, non-relativistic case.

1 QCD Lagrangian and Equations of Motion

The QCD Lagrangian density is [1] ($\hbar = c = 1$)

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^{(a)\mu\nu}F_{\mu\nu}^{(a)} + \sum_q \bar{\psi}_q^i (\mathrm{i}\gamma^\mu (D_\mu)_{ij} - m_q \delta_{ij}) \psi_q^j \quad (1-1)$$

where

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} - g_s f_{abc} A_\mu^{(b)} A_\nu^{(c)}, \quad (1-2)$$

and

$$(D_\mu)_{ij} = \delta_{ij} \partial_\mu + \mathrm{i} g_s t_{ij}^{(a)} A_\mu^{(a)} \quad \text{or} \quad \mathbf{D}_\mu = \mathbf{I}_{3 \times 3} \partial_\mu + \mathrm{i} g_s A_\mu^{(a)} \mathbf{t}^{(a)}. \quad (1-3)$$

The QCD coupling constant is g_s , f_{abc} are the structure constants, and

$$t_{ij}^{(a)} = \frac{1}{2} \lambda_{ij}^{(a)}, \quad (1-4)$$

where the λ s are the Gell-Mann matrices. As usual, repeated indices are summed over, with the colour indices $i, j = 1, 2, 3$, the flavour indices $q = d, u, s, c, b, t$, and $a, b, c = 1, \dots, 8$ for the gluon fields. Gluon indices will be usually indicated by round brackets to avoid confusion with vector indices; eg. $A_\mu^{(a)}$ or $A^{(a)\mu}$.

It is convenient to write

$$F_{\mu\nu}^{(a)} = \mathcal{F}_{\mu\nu}^{(a)} + G_{\mu\nu}^{(a)} \quad (1-5)$$

where

$$\mathcal{F}_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} \quad (1-6)$$

is the free gluon field tensor and

$$G_{\mu\nu}^{(a)} = -g_s f_{abc} A_\mu^{(b)} A_\nu^{(c)} \quad (1-7)$$

is the non-Abelian part.

Upon substituting (1-3) into (1-1), the Lagrangian density (1-1) can be written as

$$\mathcal{L}_{QCD} = \mathcal{L}_\psi + \mathcal{L}_\mathcal{F} + \mathcal{L}_{I_1} + \mathcal{L}_{I_3} + \mathcal{L}_{I_4} \quad (1-8)$$

where

$$\mathcal{L}_\psi = \sum_q \bar{\psi}_q^j (\mathrm{i} \gamma^\mu \partial_\mu - m_q) \psi_q^j \quad (1-9)$$

$$\mathcal{L}_\mathcal{F} = -\frac{1}{4} \mathcal{F}^{(a)\mu\nu} \mathcal{F}_{\mu\nu}^{(a)}, \quad (1-10)$$

$$\mathcal{L}_{I_1} = -j_\psi^{(a)\mu} A_\mu^{(a)} \quad \text{where} \quad j_\psi^{(a)\mu} = g_s \sum_q \bar{\psi}_q^i t_{ij}^{(a)} \gamma^\mu \psi_q^j, \quad (1-11)$$

$$\mathcal{L}_{I_3} = -\frac{1}{2} \mathcal{F}^{(a)\mu\nu} G_{\mu\nu}^{(a)} = \frac{1}{2} g_s f_{abc} \left(\partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} \right) A_{(b)}^\mu A_{(c)}^\nu = g_s f_{abc} A_{(a)}^\mu A_{(b)}^\nu \partial_\mu A_{(c)}^\nu, \quad (1-12)$$

$$\mathcal{L}_{I_4} = -\frac{1}{4} G^{(a)\mu\nu} G_{\mu\nu}^{(a)} = -\frac{1}{4} g_s^2 f_{abc} f_{ade} A_\mu^{(b)} A_\nu^{(c)} A_{(d)}^\mu A_{(e)}^\nu. \quad (1-13)$$

For the quark equations of motion, we need

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}_q^i} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \bar{\psi}_{q,\mu}^i} \right) = 0 \quad (1-14)$$

and using (1-9) and (1-11) we acquire the colour Dirac equations:

$$(\mathrm{i} \gamma^\mu \partial_\mu - m_q) \psi_q^i - g_s t_{ij}^{(a)} A^{(a)} \psi_q^j = 0. \quad (1-15)$$

Equations (1-15) can be written as

$$\mathrm{i} \gamma^\mu (D_\mu)_{ij} \psi_q^j - m_q \psi_q^i = 0, \quad (1-16)$$

or, in matrix notation,

$$\mathrm{i} \gamma^\mu \mathbf{D}_\mu \Psi_q - m_q \Psi_q = 0, \quad (1-17)$$

where $\mathbf{D}_\mu = [(D_\mu)_{ij}]$ is defined in equation (1-3) and $\Psi_q^T = [\psi_q^1, \psi_q^2, \psi_q^3]$ (T stands for transpose).

The equation of motion for the gluon fields is

$$\mathbf{D}^\mu F_{\mu\nu}^{(a)} = g_s \sum_q \bar{\Psi}_q \mathbf{t}^{(a)} \gamma_\nu \Psi_q, \quad (1-18)$$

where $\mathbf{t}^{(a)} = [t_{ij}^{(a)}]$ (cf. ref. [2]). In component form, eq. (1-18) is (cf. ref. [3])

$$\partial^\mu \mathcal{F}_{\mu\nu}^{(a)} = \partial_\mu \partial^\mu A_\nu^{(a)}(x) - \partial_\nu \partial^\mu A_\mu^{(a)}(x) = j_\nu^{(a)}(x) \quad (1-19)$$

where

$$j_\nu^{(a)}(x) = j_\psi^{(a)\nu}(x) + j_g^{(a)\nu}(x) + j_{g^2}^{(a)\nu}(x), \quad (1-20)$$

and

$$j_\psi^{(a)\nu} = \sum_q g_s \bar{\psi}_q^i \gamma^\nu t_{ij}^{(a)} \psi_q^j, \quad (1-21)$$

$$j_g^{(a)\nu} = g_s f_{abc} (A_{(c)}^\nu \partial_\mu A_{(b)}^\mu + 2 A_{(b)}^\mu \partial_\mu A_{(c)}^\nu + A_{(c)}^\mu \partial^\nu A_{(b)}^\mu), \quad (1-22)$$

$$j_{g^2}^{(a)\nu} = g_s^2 f_{abc} f_{bde} A_\mu^{(c)} A_{(d)}^\mu A_{(e)}^\nu. \quad (1-23)$$

2 Reformulation

For the study of inter-quark interactions (and subsequently for the study of the properties of mesons and baryons) it is convenient to use a (formal) solution of the gluon equations of motion (1-19) to reformulate the Lagrangian, and thus the action, of QCD, so that the gluon propagator appears directly in the interaction terms. Such reformulation has been shown to be useful for the study of inter-particle forces in scalar theory with a nonlinear mediating field [4, 5].

The formal solution of (1-19) involves the use of the symmetric Green function of that equation, and this requires a choice of gauge. We shall use the Lorentz gauge, $\partial_\mu A^{\mu(a)}(x) = 0$, whereupon the “glue” equation (1-19) can be rewritten as an integral equation,

$$A^{\mu(a)}(x) = \int d^4x' D(x-x') j^{\mu(a)}(x'), \quad (2-1)$$

where $D(x-x')$ is a Green function defined by

$$\partial_\nu \partial^\nu D(x-x') = \delta^4(x-x'). \quad (2-2)$$

The Green function can be written as

$$D(x-x') = \int \frac{d^4k}{(2\pi)^4} D(k) e^{-ik \cdot (x-x')}, \quad (2-3)$$

where $D(k) = \mathcal{P}/k^2$ is the momentum-space representation of the Green function. (Note: We use the notation that x, k stand for 4-vectors, i.e. $x = (t, \mathbf{x})$ and $k = (k^0, \mathbf{k})$, etc.)

We have not included the free-gluon solution of the homogeneous equation (1-19) in (2-1), since free gluons do not arise and so free-gluon solutions will play no role in the present considerations.

The expression (2-1) is only a formal solution of (1-19) (in the Lorentz gauge), because the components $j_g^{\nu(a)} + j_{g^2}^{\nu(a)}$ of the current $j^{\nu(a)}$ depend on $A^{\mu(a)}$. Unfortunately, it is not possible to obtain an explicit, closed-form solution of the non-linear equation (2-1) (i.e. of (1-19)) for $A^{\mu(a)}$ in terms of the quark fields ψ_q^i (at least we do not know how to do so). Thus, one must resort to approximation methods.

Equation (2-1) can be solved as an iterative series for $A^{\mu(a)}$ (cf. ref. [6]). The first order term in this sequence is just the expression (2-1), but with $j^{\mu(a)}$ replaced by $j_\psi^{\mu(a)}$ only, that is

$$A_{(1)}^{\mu(a)}(x) = \int d^4x' D(x-x') j_\psi^{\mu(a)}(x') = \int d^4x' D(x-x') \sum_q g_s \bar{\psi}_q^i(x') \gamma^\nu t_{ij}^{(a)} \psi_q^j(x'). \quad (2-4)$$

The index (1) in (2-4), as in all that follows, indicates that it is a first-order iterative expression. We note that the gluon fields are expressed here explicitly in terms of the quark fields ψ_q^i and the propagator D only.

Correspondingly, to first order, this modifies the Lagrangian density (1-8), to

$$\mathcal{L}_1 = \mathcal{L}_\psi - \frac{1}{2} j_\psi^{\mu(a)} A_{(1)\mu}^{(a)} + \mathcal{L}_{I_3} \left(A_{(1)}^{\nu(b)} \right) + \mathcal{L}_{I_4} \left(A_{(1)}^{\nu(b)} \right), \quad (2-5)$$

where, in (2-5), $A_{(1)}^{\nu(b)}$ is as given in (2-4).

In obtaining \mathcal{L}_1 , eq. (2-5), we have used the fact that, in the Lorentz gauge,

$$\mathcal{L}_\mathcal{F} = -\frac{1}{2} \partial_\mu A_\nu^{(a)} \partial^\mu A^{\nu(a)} \simeq \frac{1}{2} A_\nu^{(a)} \partial_\mu \partial^\mu A^{\nu(a)}, \quad (2-6)$$

where \simeq means equality modulo surface terms. Then, because $\partial_\mu \partial^\mu A^{\nu(a)} = j_\psi^{\nu(a)}$ in first order, this means that $\mathcal{L}_\mathcal{F} = \frac{1}{2} A_\nu^{(a)} j_\psi^{\nu(a)}$ (in first order). Thus, in light of (2-4), the reformulated Lagrangian

density \mathcal{L}_1 (eq. (2-5)), and so the corresponding Hamiltonian and the action, is a functional of the gluon Green function $D(x - x')$ and the quark fields ψ_q^i only. We shall refer to the theory based the Lagrangian density \mathcal{L}_1 as the “reduced model”.

At this point we might mention that in the QED (or $SU(1)$) case, the non-Abelian terms \mathcal{L}_{I_3} and \mathcal{L}_{I_4} do not arise, and \mathcal{L}_1 corresponds to the reduced QED Lagrangian. This reduced QED Lagrangian was used previously (in the quantized Hamiltonian formalism) to derive relativistic few-fermion equations and, from them, the relativistic energy spectra for all bound states of positronium [7], muonium [8] and negative positronium and muonium ions [9]. The results are exact to $O(\alpha^4)$, including, for positronium, the field-theoretic virtual annihilation correction.

3 Analysis of the inter-quark interactions in the reduced model including the static limit

We denote the action by $\mathcal{S} = \int d^4x \mathcal{L}(x)$ and shall consider, in turn, each of the interaction terms of the action in the reduced model. Thus, the term corresponding to \mathcal{L}_{I_1} , eq. (1-11), is

$$\mathcal{S}_{I_1}^{(1)} = \int d^4x \left(-\frac{1}{2} j_\psi^{\mu(a)}(x) A_{(1)\mu}^{(a)}(x) \right) = -\frac{1}{2} \int d^4x d^4x' j_\psi^{\mu(a)}(x) D(x - x') j_{\psi\mu}^{(a)}(x'), \quad (3-1)$$

where the quark-field current $j_\psi^{\mu(a)}(x)$ is given in eq. (1-11) (also in (1-21)). Since $j_\psi^{\mu(a)} \propto g_s$, we see that $\mathcal{S}_{I_1}^{(1)}$ corresponds to an energy contribution of $O(g_s^2)$. We also see that this interaction term describes the quark currents interacting via the gluon Green function D , which, in the quantised theory, would correspond to a one-gluon exchange inter-quark interaction.

To understand the physical content of the interactions corresponding to $\mathcal{S}_{I_k}^{(1)}$ ($k = 1, 2, 3$), it is useful to consider the case of static sources in the non-relativistic limit, as was done for the scalar model [4, 5]. For the static case, $j_\psi^{\mu(a)}(x) = j_\psi^{\mu(a)}(\mathbf{x})$, hence the static version of $\mathcal{S}_{I_1}^{(1)}$, eq. (3-1), is

$$\mathcal{S}_{I_1}^{(1)} = -\frac{1}{2} \int dt dt' d^3x d^3x' j_\psi^{\mu(a)}(\mathbf{x}) D(x - x') j_{\psi\mu}^{(a)}(\mathbf{x}'). \quad (3-2)$$

Since

$$D(x - x') = \frac{1}{\pi} \delta((x - x')^2) = \frac{1}{8\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} [\delta(t - t' - |\mathbf{x} - \mathbf{x}'|) + \delta(t - t' + |\mathbf{x} - \mathbf{x}'|)] \quad (3-3)$$

it follows that

$$\int dt' D(x - x') = \frac{1}{8\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \int dt' [\delta(t - t' - |\mathbf{x} - \mathbf{x}'|) + \delta(t - t' + |\mathbf{x} - \mathbf{x}'|)] = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \quad (3-4)$$

Thus, in the static case, $\mathcal{S}_{I_1}^{(1)}$, eq. (3-1), can be written as

$$\mathcal{S}_{I_1}^{(1)} = \int dt L_{I_1}^{(1)} = - \int dt H_{I_1}^{(1)} \quad \text{where} \quad H_{I_1}^{(1)} = \frac{1}{2} \int d^3x d^3x' j_\psi^{\mu(a)}(\mathbf{x}) \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}'|} j_{\psi\mu}^{(a)}(\mathbf{x}') \quad (3-5)$$

It is clear from (3-5) that $H_{I_1}^{(1)}$ corresponds to the two-point potential energy function

$$V(\mathbf{x}_1, \mathbf{x}_2) = \frac{g_s^2}{4\pi} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} = \frac{g_s^2}{4\pi} \frac{1}{x_{12}}, \quad (3-6)$$

which is the non-relativistic limit of the one-gluon exchange interaction (in coordinate representation). It reflects the $O(g_s^2)$ “Coulombic” contribution to the interquark potential. Note that it depends only on the distance, x_{12} , between the points \mathbf{x}_1 and \mathbf{x}_2 , as expected.

Similarly, the component of the action corresponding to \mathcal{L}_{I_3} , eq. (1-12), is, in first order,

$$\mathcal{S}_{I_3}^{(1)} = \int d^4x \mathcal{L}_{I_3}^{(1)}(x) = -g_s f_{abc} \int d^4x A_{(1)\nu}^{(a)}(x) A_{(1)}^{(b)\mu}(x) \partial_\mu A_{(1)}^{(c)\nu}(x). \quad (3-7)$$

This term corresponds to an $O(g_s^4)$ contribution to the energy. The Green function D appears in degree 3 in this expression, which reflects the 3-gluon vertex interaction corresponding to the “degree 3 in $A_\mu^{(a)}$ ” form of $\mathcal{L}_{I_3}^{(1)}$.

In the static case, $A_{(1)\mu}^{(a)}(x) = \int d^4x' D(x-x') j_{\psi\mu}^{(a)}(x')$ becomes

$$A_{(1)\mu}^{(a)}(\mathbf{x}) = \int d^4x' D(x-x') j_{\psi\mu}^{(a)}(\mathbf{x}') = \int d^3x' j_{\psi\mu}^{(a)}(\mathbf{x}') \int dt' D(x-x') = \int \frac{d^3x'}{4\pi} \frac{j_{\psi\mu}^{(a)}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \quad (3-8)$$

where we have used (3-4), and thus

$$\partial_t A_{(1)\nu}^{(a)}(\mathbf{x}) = 0, \quad \partial_k A_{(1)\nu}^{(a)}(\mathbf{x}) = \int \frac{d^3x'}{4\pi} j_{\psi\nu}^{(a)}(\mathbf{x}') \frac{\partial}{\partial x^k} \frac{1}{|\mathbf{x}-\mathbf{x}'|} = - \int \frac{d^3x'}{4\pi} j_{\psi\nu}^{(a)}(\mathbf{x}') \frac{x_k - x'_k}{|\mathbf{x}-\mathbf{x}'|^3}, \quad (3-9)$$

where $k = 1, 2, 3$. Therefore, in the static case,

$$\mathcal{S}_{I_3}^{(1)} = - \int dt H_{I_3}^{(1)} = -g_s f_{abc} \int d^4x A_{(1)\nu}^{(a)}(\mathbf{x}) A_{(1)}^{(b)k}(\mathbf{x}) \partial_k A_{(1)}^{(c)\nu}(\mathbf{x}). \quad (3-10)$$

Thus, by (3-8) and (3-9),

$$\begin{aligned} H_{I_3}^{(1)} &= -\frac{g_s}{(4\pi)^3} f_{abc} \int d^3x \left[\int d^3x_1 \frac{j_{\psi}^{(a)\nu}(\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|} \right] \left[\int d^3x_2 \frac{j_{\psi k}^{(b)}(\mathbf{x}_2)}{|\mathbf{x}-\mathbf{x}_2|} \right] \left[\int d^3x_3 j_{\psi\nu}^{(c)}(\mathbf{x}_3) \frac{x^k - x_3^k}{|\mathbf{x}-\mathbf{x}_3|} \right] \\ &= -\frac{g_s}{(4\pi)^3} f_{abc} \int d^3x_1 d^3x_2 d^3x_3 j_{\psi\nu}^{(a)}(\mathbf{x}_1) j_{\psi}^{(b)k}(\mathbf{x}_2) j_{\psi}^{(c)\nu}(\mathbf{x}_3) U_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \end{aligned} \quad (3-11)$$

where

$$U_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \int d^3x \frac{x^k - x_3^k}{|\mathbf{x}-\mathbf{x}_1||\mathbf{x}-\mathbf{x}_2||\mathbf{x}-\mathbf{x}_3|^3} = \frac{\partial}{\partial x_3^k} U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \quad (3-12)$$

and

$$U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = - \int \frac{d^3x}{|\mathbf{x}-\mathbf{x}_1||\mathbf{x}-\mathbf{x}_2||\mathbf{x}-\mathbf{x}_3|} = - \int \frac{d^3v}{|\mathbf{v}||\mathbf{v}-\mathbf{x}_{21}||\mathbf{v}-\mathbf{x}_{31}|}; \quad (3-13)$$

here $\mathbf{x}_{mn} = -\mathbf{x}_{nm} \equiv \mathbf{x}_m - \mathbf{x}_n$. Note that the integral in (3-13) is divergent and so must be regularised (this is discussed in ref. [5]). However the derivatives (3-12) are well-behaved (finite).

We see that the interaction term (3-11) corresponds to a three-point potential,

$$V_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = -\frac{g_s^4}{(4\pi)^3} U_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = -\frac{g_s^4}{(4\pi)^3} \frac{\partial}{\partial x_3^k} U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \quad k = 1, 2, 3, \quad (3-14)$$

which is an $O(g_s^4)$ first-iterative-order “correction” to the two-point Coulombic interaction $V(\mathbf{x}_1, \mathbf{x}_2)$ given in eq. (3-6).

Lastly, the component of the action corresponding to \mathcal{L}_{I_4} , eq. (1-13), is, in first-iterative-order

$$\mathcal{S}_{I_4}^{(1)} = \int d^4x \mathcal{L}_{I_4}(x) = -\frac{1}{4} g_s^2 f_{abc} f_{ade} \int d^4x A_{(1)\mu}^{(b)}(x) A_{(1)\nu}^{(c)}(x) A_{(1)}^{(d)\mu}(x) A_{(1)}^{(e)\nu}(x), \quad (3-15)$$

which is of $O(g_s^6)$ and of degree 4 in D , corresponding to a four gluon interaction vertex. In the static case, this term becomes

$$\mathcal{S}_{I_4}^{(1)} = -\frac{1}{4}g_s^2 f_{abc} f_{ade} \int d^4x A_{(1)\mu}^{(b)}(\mathbf{x}) A_{(1)\nu}^{(c)}(\mathbf{x}) A_{(1)}^{(d)\mu}(\mathbf{x}) A_{(1)}^{(e)\nu}(\mathbf{x}) = -\int dt H_{I_4}^{(1)}, \quad (3-16)$$

where, by (3-8),

$$\begin{aligned} H_{I_4}^{(1)} &= \frac{1}{4}g_s^2 f_{abc} f_{ade} \int d^3x A_{(1)\mu}^{(b)}(\mathbf{x}) A_{(1)\nu}^{(c)}(\mathbf{x}) A_{(1)}^{(d)\mu}(\mathbf{x}) A_{(1)}^{(e)\nu}(\mathbf{x}) \\ &= \frac{g_s^2}{4(4\pi)^4} f_{abc} f_{ade} \int d^3x_1 d^3x_2 d^3x_3 d^3x_4 j_{\psi\mu}^{(b)}(\mathbf{x}_1) j_{\psi\nu}^{(c)}(\mathbf{x}_2) j_{\psi}^{(d)\mu}(\mathbf{x}_3) j_{\psi}^{(e)\nu}(\mathbf{x}_4) U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4), \end{aligned} \quad (3-17)$$

with

$$U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \int \frac{d^3x}{|\mathbf{x} - \mathbf{x}_1| |\mathbf{x} - \mathbf{x}_2| |\mathbf{x} - \mathbf{x}_3| |\mathbf{x} - \mathbf{x}_4|} = \int \frac{d^3v}{|\mathbf{v}| |\mathbf{v} - \mathbf{x}_{21}| |\mathbf{v} - \mathbf{x}_{31}| |\mathbf{v} - \mathbf{x}_{41}|}. \quad (3-18)$$

If we include the coupling constants (and related factors) the corresponding potential-energy function is

$$V_{I_4}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \frac{1}{4} \frac{g_s^6}{(4\pi)^4} U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4). \quad (3-19)$$

This is an $O(g_s^6)$ first-order-iterative four-point potential correction to the Coulombic two-point potential (3-6).

Unfortunately, the integrals (3-12), (3-13) and (3-18) that define the three and four point potentials cannot, in general, be evaluated explicitly, that is they cannot be expressed in terms of common analytic functions (at least we do not know how to do so). Nevertheless various general properties of these first-iterative-order non-Abelian corrections to the Coulombic inter-quark potential can be readily established, and analytical expressions can be obtained for particular situations.

The general properties and representations of U_k and $U^{(4)}$ are presented and discussed in the Appendix, where it is shown, inter alia, that

1). The “potentials” $U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ are, in fact, functions of the distances $x_{mn} = |\mathbf{x}_m - \mathbf{x}_n|$ only (as might be expected of a closed system).

2). The evaluation of $U_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ can be reduced to the computation of two single quadratures and that of $U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ to a double quadrature, which must be done numerically.

As mentioned, the three and four point functions $U^{(3)}$ and $U^{(4)}$ can be evaluated analytically for some particular cases. Thus, when all three distances are equal, *i.e.* $x_{12} = x_{13} = x_{23} = r$, the (regularised) three-point function of eq. (3-14) becomes $U^{(3)}(r) = 4\pi \ln(r/a)$, where a is an arbitrary distance scale [4, 5]. Similarly, when points \mathbf{x}_2 and \mathbf{x}_3 are coincident, *i.e.* $x_{23} = 0$ and $x_{12} = x_{13}$, $U^{(3)}(x_{12}) = 4\pi \ln(x_{12}/a)$, and so

$$U_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{x}_2) \equiv \partial U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) / \partial x_3^k \Big|_{\mathbf{x}_3 = \mathbf{x}_2} = \frac{1}{2} \partial U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) / \partial x_2^k = -2\pi \frac{(\mathbf{x}_1 - \mathbf{x}_2)_k}{|\mathbf{x}_1 - \mathbf{x}_2|^2}.$$

This shows that the corresponding correction to the Coulombic one-gluon exchange potential (3-6) due to the “cubic” interaction term (3-12), in the non-relativistic limit, is of the form (cf. (3-14))

$$V_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 = \mathbf{x}_2) = \frac{g_s^4}{2(4\pi)^2} \frac{(\mathbf{x}_1 - \mathbf{x}_2)_k}{|\mathbf{x}_1 - \mathbf{x}_2|^2}. \quad (3-20)$$

Similarly, for the particular case $\mathbf{x}_1 = \mathbf{x}_3$ and $\mathbf{x}_2 = \mathbf{x}_4$, there is only one distance, x_{12} , between the two pairs of coincident points, and the four-point potential function $U^{(4)}$, eq. (3-18), and so V_{I_4} , eq. (3-19), can be evaluated explicitly:

$$U^{(4)}(x_{12}) = \int \frac{d^3v}{|\mathbf{v}|^2 |\mathbf{v} + \mathbf{x}_{12}|^2} = \frac{\pi^3}{x_{12}}, \quad \text{hence} \quad V_{I_4}(x_{12}) = \frac{g_s^6}{4^5 \pi} \frac{1}{x_{12}}. \quad (3-21)$$

Although the expressions (3-20) and (3-21) are only segments of the three and four point potentials $V_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, eq'n (3-14), and $V_{I_4}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, eq'n (3-19), they suggest that their behaviour is Coulomb-like in general. Note that these corrections are $O(g_s^4)$ and $O(g_s^6)$ respectively.

4 Concluding remarks

We have used an approximate, iterative solution of the non-linear classical equations of motion of QCD to derive expressions for the interaction terms corresponding to the non-Abelian terms (1-12) and (1-13) of the QCD action. In first iterative order, cf. equation (2-4), these turn out to be expressions involving products of three and four one-gluon exchange Green functions, corresponding to three- and four-gluon interaction vertices (cf. eq'ns (3-7) and (3-15) respectively).

We have examined these non-Abelian terms in the static, non-relativistic limit and found them to be three- and four-point static potentials, (3-14) and (3-19), that depend on the inter point coordinates only. Although we could not express these three- and four-point potentials in terms of common analytic functions in general, we could do so for some restricted sections and these indicate that the potentials $V_k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $V^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ are Coulomb-like in general.

The derived three- and four-point cluster corrections together with the one-gluon exchange interaction could be used as a short-range contribution in potential models of baryons, tetra-quarks etc.

The quantized theory, based on the reduced Lagrangian (2-5), can be used to derive relativistic few-quark equations, as was done for the scalar theory with non-linear mediating fields [4, 5]. This shall be left for future work.

Appendix. Properties and evaluation of the three- and four-point potentials

Components of the three-point potential (3-12) form the vector potential:

$$U(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{\partial}{\partial \mathbf{x}_3} U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \quad (\text{A-1})$$

where the function $U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ (see eq.(3-13)) is studied in [4]. Particularly useful is the following representation this function:

$$\begin{aligned} U^{(3)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= -\frac{1}{\pi^{3/2}} \int d^3 k \int d^3 x \, e^{-k_1^2(\mathbf{x}-\mathbf{x}_1)^2 - k_2^2(\mathbf{x}-\mathbf{x}_2)^2 - k_3^2(\mathbf{x}-\mathbf{x}_3)^2} \\ &= -\int \frac{d^3 k}{k^3} e^{-(k_1^2 k_2^2 x_{12}^2 + k_2^2 k_3^2 x_{23}^2 + k_1^2 k_3^2 x_{13}^2)/k^2} \\ &= -\int d^2 \hat{k} \int_0^\infty \frac{dk}{k} e^{-(\hat{k}_1^2 \hat{k}_2^2 x_{12}^2 + \hat{k}_2^2 \hat{k}_3^2 x_{23}^2 + \hat{k}_1^2 \hat{k}_3^2 x_{13}^2)k^2}, \end{aligned} \quad (\text{A-2})$$

where $x_{mn} \equiv |\mathbf{x}_m - \mathbf{x}_n|$ ($m, n = 1, 2, 3$), $\hat{k} = \mathbf{k}/k$, $k = |\mathbf{k}| \equiv \sqrt{k_1^2 + k_2^2 + k_3^2}$ and $\int d^2 \hat{k}$ denotes integration over the unit sphere in 3D k -space. The integral (A-2) (as well as (3-13)) is divergent and needs to be regularized. One way is to split it into two terms: $U^{(3)} = \tilde{U} + U_0$, where $\tilde{U}(x_{12}, x_{23}, x_{13})$ is a finite function of three scalar arguments and $U_0 = U(a, b, c)$ is an “infinite constant” (a, b, c are arbitrary constants) [4].

Inserting (A-2) into r.h.s. of (A-1) discards the infinite constant U_0 and yields the formula:

$$\begin{aligned} U &= \frac{\partial U^{(3)}}{\partial \mathbf{x}_3} = -\frac{\partial}{\partial \mathbf{x}_3} \int d^2 \hat{k} \int_0^\infty \frac{dk}{k} e^{-(\hat{k}_1^2 \hat{k}_2^2 x_{12}^2 + \hat{k}_2^2 \hat{k}_3^2 x_{23}^2 + \hat{k}_3^2 \hat{k}_1^2 x_{31}^2) k^2} \\ &= 2 \int d^2 \hat{k} \hat{k}_3^2 \left(\hat{k}_1^2 \mathbf{x}_{31} + \hat{k}_2^2 \mathbf{x}_{32} \right) \int_0^\infty dk k e^{-(\hat{k}_1^2 \hat{k}_2^2 x_{12}^2 + \dots) k^2} \equiv \mathbf{x}_{31} I_1 + \mathbf{x}_{32} I_2, \end{aligned} \quad (\text{A-3})$$

where

$$I_n = \int \frac{d^2 \hat{k} \hat{k}_n^2 \hat{k}_3^2}{\hat{k}_1^2 \hat{k}_2^2 x_{12}^2 + \hat{k}_2^2 \hat{k}_3^2 x_{23}^2 + \hat{k}_1^2 \hat{k}_3^2 x_{13}^2}, \quad n = 1, 2. \quad (\text{A-4})$$

Next, we introduce angular variables $\{\vartheta, \varphi\}$ on the unit sphere in 3D k -space, so that $\hat{k}_1 = \sin \vartheta \cos \varphi$, $\hat{k}_2 = \sin \vartheta \sin \varphi$, $\hat{k}_3 = \cos \vartheta$. Then

$$I_1 = \int_0^{2\pi} d\varphi \cos^2 \varphi J, \quad I_2 = \int_0^{2\pi} d\varphi \sin^2 \varphi J, \quad (\text{A-5})$$

where

$$\begin{aligned} J &= \int_0^\pi \frac{d\vartheta \sin \vartheta \cos^2 \vartheta}{(x_{12} \sin \vartheta \cos \varphi \sin \varphi)^2 + \cos^2 \vartheta (x_{13}^2 \cos^2 \varphi + x_{23}^2 \sin^2 \varphi)} \\ &= \frac{8}{x_{12}^2 R^2} \left[1 - \frac{\sin 2\varphi}{R} \arctan \frac{R}{\sin 2\varphi} \right] \end{aligned} \quad (\text{A-6})$$

$$\begin{aligned} \text{and } R &= \sqrt{(\cos 2\varphi + \xi)^2 + \eta^2}, \\ \xi &= \frac{x_{13}^2 - x_{23}^2}{x_{12}^2}, \quad \eta^2 = \frac{[(x_{13} + x_{23})^2 - x_{12}^2][x_{12}^2 - (x_{13} - x_{23})^2]}{x_{12}^4}. \end{aligned} \quad (\text{A-7})$$

Inserting (A-6) into (A-5) and using the integration variable $s = \cos 2\varphi$ yields the quadratures:

$$\begin{aligned} I_1 &= \frac{8}{x_{12}^2} \int_{-1}^1 \frac{ds}{R^2} \left[\sqrt{\frac{1+s}{1-s}} - \frac{1+s}{R} \arctan \frac{R}{\sqrt{1-s^2}} \right], \\ I_2 &= \frac{8}{x_{12}^2} \int_{-1}^1 \frac{ds}{R^2} \left[\sqrt{\frac{1-s}{1+s}} - \frac{1-s}{R} \arctan \frac{R}{\sqrt{1-s^2}} \right]. \end{aligned} \quad (\text{A-8})$$

Note that for the special case $x_{23} = 0$, the expressions (A-3) to (A-8) yield the result (3-20). In the general case, that is, for arbitrary values of x_{12} , x_{13} and x_{23} , the integrals (A-8) need to be evaluated numerically.

The behavior of the three-point vector potential as a function of \mathbf{x}_3 is illustrated in figure 1. We note the following symmetry properties of the vector potential:

- translational invariance: $\mathbf{U}(\mathbf{x}_1 + \boldsymbol{\lambda}, \mathbf{x}_2 + \boldsymbol{\lambda}, \mathbf{x}_3 + \boldsymbol{\lambda}) = \mathbf{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, where $\boldsymbol{\lambda} \in \mathbb{R}^3$;
- rotational covariance: $\mathbf{U}(\mathbf{R}\mathbf{x}_1, \mathbf{R}\mathbf{x}_2, \mathbf{R}\mathbf{x}_3) = \mathbf{R}\mathbf{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, where $\mathbf{R} \in \text{SO}(3)$;
- partial permutational invariance: $\mathbf{U}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3) = \mathbf{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$;
- scaling transformation: $\mathbf{U}(\lambda\mathbf{x}_1, \lambda\mathbf{x}_2, \lambda\mathbf{x}_3) = \lambda^{-1}\mathbf{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, where $\lambda \in \mathbb{R}_+$.

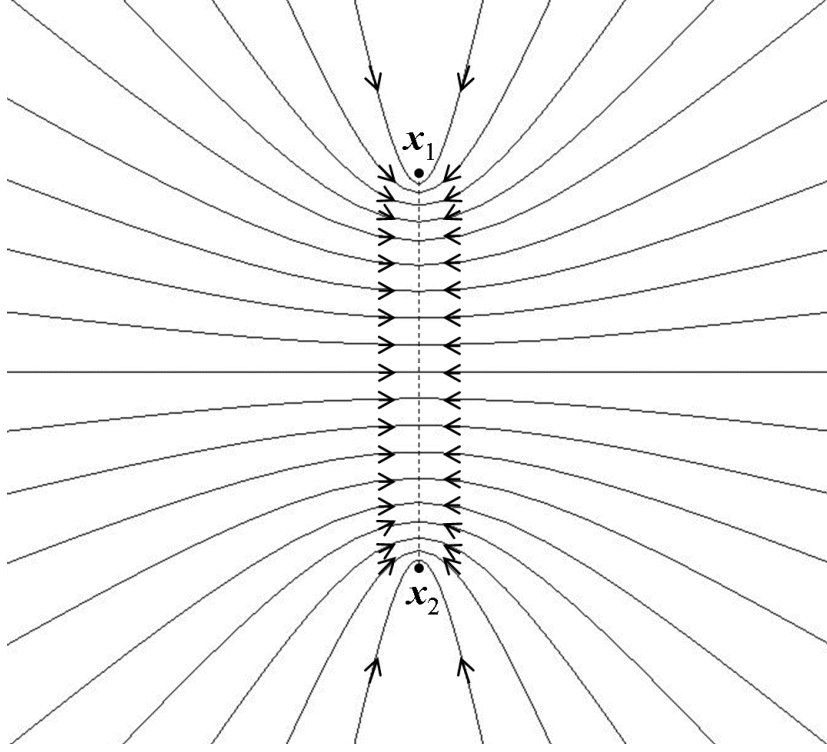


Figure 1: Two-dimensional section of the vector potential $\mathbf{U}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ as a function of $\mathbf{x}_3 = \mathbf{r}$ for fixed \mathbf{x}_1 and \mathbf{x}_2 . The arrows indicate the direction of the vector field $\mathbf{U}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{r})$. Note that it is invariant with respect rotations about the axis $\mathbf{x}_1 - \mathbf{x}_2$. The dashed line between \mathbf{x}_1 and \mathbf{x}_2 corresponds to $\mathbf{U} = 0$.

These properties follow from the properties of the three-point scalar potential (3-13) stated in [4, 5]. We also note that the scaling transformation of the potential \mathbf{U} reflects its Coulomb-like behaviour.

The four-point scalar potential (3-18) can be treated in similar manner:

$$\begin{aligned}
 U^{(4)}(\mathbf{x}_1, \dots, \mathbf{x}_4) &= \int \frac{d^3x}{|\mathbf{x} - \mathbf{x}_1| \cdots |\mathbf{x} - \mathbf{x}_4|} = \frac{1}{\pi^2} \int d^4k \int d^3x e^{-k_1^2(\mathbf{x} - \mathbf{x}_1)^2 - \cdots - k_4^2(\mathbf{x} - \mathbf{x}_4)^2} \\
 &= \frac{1}{\sqrt{\pi}} \int d^3\hat{k} \int_0^\infty dk e^{-X^2 k^2} = \int \frac{d^3\hat{k}}{\sqrt{X^2}},
 \end{aligned} \tag{A-9}$$

$$\text{where } X^2 \equiv \hat{k}_1^2 \hat{k}_2^2 x_{12}^2 + \hat{k}_1^2 \hat{k}_3^2 x_{13}^2 + \hat{k}_1^2 \hat{k}_4^2 x_{14}^2 + \hat{k}_2^2 \hat{k}_3^2 x_{23}^2 + \hat{k}_2^2 \hat{k}_4^2 x_{24}^2 + \hat{k}_3^2 \hat{k}_4^2 x_{34}^2,$$

$\hat{k}_n = k_n/k$ ($n = 1, \dots, 4$), $k = \sqrt{k_1^2 + \cdots + k_4^2}$, and $\int d^3\hat{k}$ denotes an integration over a unit hyper-sphere in 4D k -space.

Next, we introduce angular variables $\{\chi, \vartheta, \varphi\}$ on the unit hyper-sphere in 4D k -space, so that $\hat{k}_1 = \sin \chi \sin \vartheta \cos \varphi$, $\hat{k}_2 = \sin \chi \sin \vartheta \sin \varphi$, $\hat{k}_3 = \sin \chi \cos \vartheta$, $\hat{k}_4 = \cos \chi$, and $\int d^3\hat{k} =$

$\int_0^{2\pi} d\varphi \int_0^\pi \sin \vartheta d\vartheta \int_0^\pi \sin^2 \chi d\chi$. Then using the integration variables $u = \cos \chi$, $v = \cos \vartheta$, $w = \cos 2\varphi$ reduces the integral (A-9) to the form:

$$U^{(4)}(x_{12}, \dots, x_{34}) = 4 \int_{-1}^1 \frac{dw}{\sqrt{1-w^2}} \int_0^1 dv I, \quad (\text{A-10})$$

$$\begin{aligned} \text{where } I &= \int_0^1 \frac{du}{\sqrt{A^2 + u^2 B^2}} = \frac{1}{B} \ln \left(\frac{\sqrt{A^2 + B^2} + B}{A} \right), \\ A^2 &= \left\{ \frac{1}{4} x_{12}^2 (1 - w^2) (1 - v^2) + \frac{1}{2} [x_{13}^2 + x_{23}^2 + (x_{13}^2 - x_{23}^2) w] v^2 \right\} (1 - v^2), \\ B^2 &= \frac{1}{2} [x_{14}^2 + x_{24}^2 + (x_{14}^2 - x_{24}^2) w] (1 - v^2) + x_{34}^2 v^2 - A^2 \end{aligned}$$

This double integral can be evaluated numerically. Evaluation of (A-10) for the case $\mathbf{x}_1 = \mathbf{x}_3$ and $\mathbf{x}_2 = \mathbf{x}_4$ gives the same result as (3-21).

Finally we note, that the four-point potential (A-10) possesses the same translational invariance and scaling transformation properties as the three-point one does. Besides, it is invariant under arbitrary rotation and permutation of its arguments:

- rotational invariance: $U^{(4)}(\mathbf{R}\mathbf{x}_1, \mathbf{R}\mathbf{x}_2, \mathbf{R}\mathbf{x}_3, \mathbf{R}\mathbf{x}_4) = U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$, where $\mathbf{R} \in \text{SO}(3)$;
- complete permutational invariance:
 $U^{(4)}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4) = U^{(4)}(\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_4) = \dots = U^{(4)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$.

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